PDAE models of integrated circuits and perturbation analysis*

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SUMMARY

A model for a linear electric circuit containing semiconductors is presented. The modified nodal analysis leads to a differential algebraic equation (DAE) describing the electric network. The non-linear behaviour of the semiconductors is modelled by the drift diffusion equations. Coupling relations are defined and a sensitivity analysis concept that generalises the DAE index for finite systems to infinite ones is presented and applied to the resulting partial differential algebraic equation (PDAE). It is shown that the coupled system is of index 1 if the voltages applied to the semiconductors are low and the network without semiconductors is of index 1.

Keywords: semiconductor, partial differential algebraic equation, sensitivity analysis, index.

1. INTRODUCTION

In the development of integrated memory circuits, the modelling of semiconductors with equivalent models is getting more and more cumbersome. The decreasing spatial scales and higher frequencies lead to larger equivalent models requiring an extensive tuning effort. Therefore, it is worthwhile to replace them by partial differential equations (PDEs).

A model containing the stationary drift diffusion equations of an integrated circuit is presented. We focus on the transient behaviour and analyse the model with respect to perturbations. In [1] it was shown that, for the finite case, tractability index 1 implies perturbation index 1. A generalisation of the tractability index to infinite dimensional systems is presented and applied. We show that the resulting partial differential algebraic equation (PDAE) is of index 1, if the electric network without semiconductors is of index 1, an additional topological condition is satisfied, and the applied voltages are low. In the future we aim at investigating how the generalised tractability index

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relates to sensitivity with respect to time-dependent perturbations.

In Section 2 we model the electric network, present the drift diffusion equations and derive boundary conditions. The coupled system and the concept of the abstract differential algebraic system (ADAS) index are presented in the Sections 3 and 4. The greater part of the index proof consists of an existence proof for the linearised drift diffusion equations contained in Section 5. Thereafter, we summarise our results and give an example.

2. MODELLING INTEGRATED CIRCUITS WITH SEMICONDUCTOR PDES

We consider an RLC network with one semiconductor modelled by the drift diffusion equations. Generalisation to several semiconductors is straightforward. The circuit with $n+1$ nodes contains only diodes, linear capacitors, linear inductors, linear resistors and independent voltage and current sources. The behaviour of the controlled sources is described by given time-dependent functions, $i_s(\cdot) \in \mathbb{R}^{k_I}$ and $v_s(\cdot) \in \mathbb{R}^{k_V}$, respectively. In the modified nodal analysis (MNA) [2], Kirchoff's laws and the specific relations describing the network elements are combined in a differential algebraic equation (DAE). The unknowns are reduced to a vector $x(t) = (e(t), i_L(t), i_V(t))^T \in \mathbb{R}^{n+k_V+k_I}$ containing the node potentials, the currents through the inductors and the currents through the voltage sources. The DAE takes the form ([3], [4])

$$\begin{pmatrix} A_C C A_C^T & 0 \\ 0 & L \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} P_C & 0 & 0 \\ 0 & I_{k_L} & 0 \end{pmatrix} x \right)' + \begin{pmatrix} A_R G A_R^T & A_L & A_V \\ -A_L^T & 0 & 0 \\ A_V^T & 0 & 0 \end{pmatrix} x +$$

$$\begin{pmatrix} A_S j_S \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} A_I i_s \\ 0 \\ -v_s \end{pmatrix} = 0. \tag{1}$$

The capacitance, inductance and resistance matrices

$$C \in \mathbb{R}^{k_C \times k_C}, \qquad L \in \mathbb{R}^{k_L \times k_L}, \qquad G \in \mathbb{R}^{k_G \times k_G},$$

are symmetric positive definite. The incidence matrices

$$A_C \in \mathbb{R}^{n \times k_C}, \qquad A_L \in \mathbb{R}^{n \times k_L}, \qquad A_G \in \mathbb{R}^{n \times k_G},$$

$$A_V \in \mathbb{R}^{n \times k_V}, \qquad A_I \in \mathbb{R}^{n \times k_I}, \qquad A_S \in \mathbb{R}^{n \times k_\lambda},$$

describe the positions in the network of the capacitors, inductors, resistors, independent voltage and current sources and the semiconductor, respectively. The current through the semiconductor is denoted by j_S.

The choice of an appropriate numerical method is guided by an index classification of the DAE. The value of the index relates to the number of differentiations of the data, a numerically ill-conditioned operation, required in the solution process. Thus, the higher the index is, the greater the numerical difficulties are.

The interesting question we want to study here is the influence of time-dependent perturbance supposing the network is of index 1 when the semiconductor branch is omitted. In [4] it is shown that the circuit is of index 1 if and only if the network without the semiconductors (or networks-if removing the semiconductor branch splits the network) contains neither CV-loops nor LI-cutsets. A CV-loop is a closed path containing only capacitors and at least one controlled voltage source. A cutset is a set of branches having the property that, when the entire set is removed, the network splits into two non-connected parts, but when all elements except for an arbitrary one in the set is removed, the network remains connected. If the set contains only inductors and controlled current sources it is an LI-cutset.

Rewriting (1) as

$$A(Dx(t))' + Bx(t) + Ej_S(t) + q(t) = 0 \tag{2}$$

we find that the network without the semiconductors is of tractability index 1 if and only if $G_1 = AD + BQ_0$ is nonsingular [1]. Here, Q_0 is a projector onto ker AD and may be chosen as

$$Q_0 = \begin{pmatrix} Q_C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{k_V} \end{pmatrix}$$

with $Q_C \in \mathbb{R}^{k_C \times k_C}$ being a projector onto ker A_C^T.

A semiconductor occupying a region $\Omega \in \mathbb{R}^3$ with some doping profile $C(\cdot) \in L_2(\Omega)$ is considered. The interaction between the electrostatic potential ψ, the charge carrier densities n, p, and the current densities J_n, J_p, can be modelled by the stationary drift diffusion equations

$$\epsilon \Delta \psi = q(n - p - N), \tag{3a}$$

$$\mathrm{div} J_n = qR(n,p), \tag{3b}$$

$$\mathrm{div} J_p = -qR(n,p), \tag{3c}$$

$$J_n = q\mu_n(U_T \nabla n - n \nabla \psi), \tag{3d}$$

$$J_p = -q\mu_p(U_T \nabla p + p \nabla \psi). \tag{3e}$$

The functions $\psi,\ n,\ p,\ J_n$ and J_p depend on space, $y \in \Omega$, and time, $t \in [0, \infty)$. The implicit time dependence cannot be omitted since we are to apply time-dependent boundary conditions determined by the surrounding circuit and then to analyse the sensitivity of the coupled system with respect to time-dependent perturbations.

Close to thermal equilibrium we can assume constant temperature. Consequently, the thermal potential U_T is constant. Temperature-dependent models of integrated circuits are studied in [5]. Other constants are the elementary charge q and the electric permittivity constant ϵ.

Generally, the mobilities of the charge carriers, μ_n and μ_p, depend on the doping profile and the electric field, but, for low applied voltages it is sufficient to model them as only space(doping)-dependent non-negative functions.

The function R models the recombination and generation of charge carriers. We use the Shockley-Read-Hall recombination,

$$R(n,p) = \frac{np - n_i^2}{\tau_p(n+n_i) + \tau_n(p+n_i)}, \tag{4}$$

for two particle transition which is sufficient close to thermal equilibrium. The constants n_i, τ_n and τ_p denote the intrinsic carrier concentration in the material and the average times between generation and recombination for the charge carriers.

We consider a one-port semiconductor with metal-semiconductor contacts. The device geometry is a region $\Omega \subset \mathbb{R}^3$ with boundary segments consisting of two non-empty closed line segments $\Gamma_O = \Gamma_{O_1} \cup \Gamma_{O_2}$ with Dirichlet conditions and one non-empty insulating segment Γ_I with Neumann conditions. We also assume $\Gamma_{O_1} \cap \Gamma_{O_2} = \{\emptyset\}$. This assumption is needed in the construction of solution spaces independent of time and it only excludes semiconductor geometries with a short-circuit between the metal-semiconductor contacts.

Since the expressions for the current densities, (3d), (3e), can be inserted into the continuity equations, $(3b)$, $(3c)$, we need boundary conditions for ψ, n and p only.

At the metal-semiconductor contacts one assumes a vanishing space charge

$$n(y) - p(y) - N(y) = 0, \qquad y \in \Gamma_O, \tag{5}$$

and thermal equilibrium for the charge carriers

$$n(y)p(y) = n_i^2, \qquad y \in \Gamma_O. \tag{6}$$

From these equations the Dirichlet boundary conditions for n and p can be calculated

$$n(y) = \frac{1}{2}\left(N(y) + \sqrt{N(y)^2 + 4n_i^2}\right), \qquad y \in \Gamma_O, \tag{7a}$$

$$p(y) = \frac{1}{2}\left(-N(y) + \sqrt{N(y)^2 + 4n_i^2}\right), \qquad y \in \Gamma_O. \tag{7b}$$

For the electrostatic potential the boundary conditions at the metal-semiconductor contacts are determined by the surrounding electric network and the built-in potential ψ_{bi}, which originate from the doping profile C. The contact O_1 is connected to node

i and O_2 is connected to node j in the network. Then, we have

$$\psi(y,t) = \psi_{bi}(y) + e_i(t), \qquad y \in \Gamma_{O_1}, \tag{8a}$$

$$\psi(y,t) = \psi_{bi}(y) + e_j(t), \qquad y \in \Gamma_{O_2}. \tag{8b}$$

The built-in potential is defined in such a way that the device is in thermal equilibrium when the externally applied potentials are zero, that is, $e_i(t) = e_j(t) = 0$. The current densities vanish in thermal equilibrium, by integrating (3d) and (3e) and considering (6), we obtain

$$n_e = n_i \exp(\psi_e/U_T), \qquad p_e = n_i \exp(-\psi_e/U_T), \tag{9}$$

where n_e, p_e and ψ_e denote the corresponding quantities in equilibrium. By means of (7), we now calculate the built-in potential at the metal-semiconductor contacts and extend it to Ω,

$$\psi_{bi}(y) = U_T \ln\left(\frac{N(y) + \sqrt{N(y^2) + 4n_i}}{2n_i}\right), \qquad y \in \Omega. \tag{10}$$

For analytical reasons it is useful to obtain a divergence structure on the PDE system via transformation of carrier densities into the Slotboom variables U_1, U_2;

$$n = n_i \exp(\psi/U_T) U_1, \tag{11a}$$

$$p = n_i \exp(-\psi/U_T) U_2. \tag{11b}$$

In [6] it has been shown that the charge carrier densities are positive and it is obvious that this property remains valid when transforming into the Slotboom variables. There is a physical interpretation of this variable change. The carrier densities can be expressed by

$$n = n_i \exp(\frac{\psi - \phi_n}{U_T}), \tag{12a}$$

$$p = n_i \exp(-\frac{\psi - \phi_p}{U_T}), \tag{12b}$$

and if we assume that the Boltzmann statistics holds, we can interpret ϕ_n and ϕ_p as the quasi-Fermi potentials. The Slotboom variables are scaled exponentials of the latter ones.

Using the Slotboom variables, the relations for the current densities are transformed into

$$J_n = q\mu_n U_T n_i \exp(\psi/U_T) \nabla U_1, \tag{13a}$$

$$J_p = -q\mu_p U_T n_i \exp(-\psi/U_T)\nabla U_2. \tag{13b}$$

To be able to formulate solution spaces independent of time, we homogenise the unknowns, starting with the electrostatic potential.

If we use Kirchoff's Voltage Law with the matrix E from equation (2) and define a function $h \in C^\infty(\bar{\Omega})$ fulfilling

$$h(y) = 1, \qquad y \in \Gamma_{O_1}, \tag{14a}$$

$$h(y) = 0, \qquad y \in \Gamma_{O_2}, \tag{14b}$$

$$\nabla h(y) \cdot \nu(y) = 0, \qquad y \in \Gamma_I. \tag{14c}$$

we can merge (8) in

$$\psi(y,t) = \psi_{bi}(y) + h(y)E^T x(t) + e_j(t), \qquad y \in \Gamma_O. \tag{15}$$

The vector ν is the exterior unit normal on the boundary $\partial\Omega$. Note that, the condition $\Gamma_{O_1} \cap \Gamma_{O_2} = \{\emptyset\}$ is fulfilled for all standard semiconductor geometries and ensures the existence of a function h fulfilling (14). Now, the function

$$u_0(y,t) := \psi(y,t) - (\psi_{bi}(y) + \psi_E(y,t)), \tag{16}$$

where $\psi_E(y,t) = h(y)E^T x(t) + e_j(t)$ is introduced to shorten notation, satisfies homogeneous Dirichlet conditions at the metal-semiconductor contacts,

$$u_0(y,t) = 0, \qquad y \in \Gamma_O. \tag{17}$$

On the insulating boundary we assume a vanishing outward electric field and vanishing outward current densities,

$$\nabla\psi(y) \cdot \nu(y) = J_n(y) \cdot \nu(y) = J_p(y) \cdot \nu(y) = 0, \qquad y \in \Gamma_I. \tag{18}$$

Close to the insulating boundary the doping profiles of most semiconductors are constant in the boundary normal direction, that is, we can assume that

$$\nabla N(y) \cdot \nu(y) = 0, \qquad y \in \Gamma_I, \tag{19}$$

which yields

$$\nabla\psi_{bi}(y) \cdot \nu(y) = 0, \qquad y \in \Gamma_I. \tag{20}$$

Using (14c), (18) and (20), homogeneous Neumann conditions on the insulating boundary are obtained for the homogenised electrostatic potential,

$$\nabla u_0(y,t) \cdot \nu(y) = 0, \qquad y \in \Gamma_I. \tag{21}$$

We homogenise the Slotboom variables, U_1 and U_2. From (7), (10) and (15), we obtain the conditions

$$U_1(y,t) = \exp\big(-\frac{\psi_E(y,t)}{U_T}\big), \qquad y \in \Gamma_O \tag{22a}$$

$$U_2(y,t) = \exp\big(\frac{\psi_E(y,t)}{U_T}\big), \qquad y \in \Gamma_O. \tag{22b}$$

Hence, by defining

$$u_1(y,t) := U_1(y,t) - \exp\big(-\frac{\psi_E(y,t)}{U_T}\big), \tag{23a}$$

$$u_2(y,t) := U_2(y,t) - \exp\big(\frac{\psi_E(y,t)}{U_T}\big), \tag{23b}$$

we have homogeneous Dirichlet conditions for the variables u_1, u_2 on the metal-semiconductor contacts.

As for the insulating boundary, according to (13) the vanishing outward current densities (18), the condition (14c) and the assumption of positive charge carrier mobilities, directly yield homogeneous Neumann conditions

$$\nabla u_1(y) \cdot \nu(y) = 0, \qquad y \in \Gamma_I, \tag{24a}$$

$$\nabla u_2(y) \cdot \nu(y) = 0, \qquad y \in \Gamma_I. \tag{24b}$$

3. THE COUPLED SYSTEM

The network DAE and the semiconductor PDEs are coupled in two ways. First, the node potentials in the network appear in the boundary conditions for the electrostatic potential. This has been taken into account and stressed by incorporating the network variable vector $x(t)$ in the homogenisation process.

Second, the current flowing over the metal-semiconductor boundaries j_S must be taken into account in Kirchoff's Current Law for the network. By adding up $(3b)$ and $(3c)$ we have

$$\mathrm{div}(J_n + J_p) = 0, \qquad \forall y \in \Omega. \tag{25}$$

The divergence theorem yields

$$\int_{\Gamma_{O_1}} (J_n + J_p) \cdot \nu \; d\sigma = -\int_{\Gamma_{O_2}} (J_n + J_p) \cdot \nu \; d\sigma \tag{26}$$

and, hence we only need to evaluate j_S at Γ_{O_1},

$$j_S(t) = -\int_{\Gamma_{O_1}} (J_n + J_p) \cdot \nu \, d\sigma. \tag{27}$$

When multi-terminal semiconductors are considered, all but one boundary integral have to be calculated and the incidence matrix A_S has to be modified.

We have derived boundary conditions and coupling relations for a linear electric circuit containing a one-port semiconductor. The resulting PDAE system, whose transient behaviour we want to analyse, is of the form

$$A(Dx)' + Bx + Ej_S + q = 0, \tag{28a}$$

$$\begin{aligned} j_S = -\int_{\Gamma_{O_1}} n_i U_T \Big[& \mu_n \exp\big(\frac{u_0 + \psi_{bi} + \psi_E}{U_T}\big) \nabla\big(u_1 + \exp\big(\frac{-\psi_E}{U_T}\big)\big) \\ & - \mu_p \exp\big(\frac{-u_0 - \psi_{bi} - \psi_E}{U_T}\big) \nabla\big(u_2 + \exp\big(\frac{\psi_E}{U_T}\big)\big)\Big] \nu \, d\sigma, \end{aligned} \tag{28b}$$

$$\begin{aligned} \frac{\epsilon}{q}\Delta\big(u_0 + \psi_{bi} + \psi_E\big) = {} & n_i \exp\big(\frac{u_0 + \psi_{bi} + \psi_E}{U_T}\big)\big(u_1 + \exp\big(\frac{-\psi_E}{U_T}\big)\big), \\ & - n_i \exp\big(\frac{-u_0 - \psi_{bi} - \psi_E}{U_T}\big)\big(u_2 + \exp\big(\frac{\psi_E}{U_T}\big)\big) - N, \end{aligned} \tag{28c}$$

$$\operatorname{div}\big(\mu_n n_i U_T \exp\big(\frac{u_0 + \psi_{bi} + \psi_E}{U_T}\big) \nabla\big(u_1 + \exp\big(\frac{-\psi_E}{U_T}\big)\big)\big) = S(u_0, u_1, u_2, x), \tag{28d}$$

$$\operatorname{div}\big(\mu_p n_i U_T \exp\big(\frac{-u_0 - \psi_{bi} - \psi_E}{U_T}\big) \nabla\big(u_2 + \exp\big(\frac{\psi_E}{U_T}\big)\big)\big) = S(u_0, u_1, u_2, x), \tag{28e}$$

and is subject to an initial condition, consistent [7] with the DAE (28a),

$$x_0 = x(0), \tag{29}$$

and boundary conditions

$$u_j(y, t) = 0, \qquad y \in \Gamma_O, \quad \forall t \in [0, \infty), \tag{30a}$$

$$\nabla u_j(y, t) \cdot \nu(y) = 0 \qquad y \in \Gamma_I, \quad \forall t \in [0, \infty), \tag{30b}$$

for $j = 0, 1, 2$. The function S is the corresponding recombination-generation term for the transformed homogenised variables

$$R\big(u_0 + \psi_{bi} + \psi_E,\ n_i e^{\frac{u_0+\psi_{bi}+\psi_E}{U_T}}(u_1 + e^{\frac{-\psi_E}{U_T}}),\ n_i e^{-\frac{u_0+\psi_{bi}+\psi_E}{U_T}}(u_2 + e^{\frac{\psi_E}{U_T}})\big)$$
$$= S(u_0, u_1, u_2, x).$$

It also depends on the network variables x since the applied external potential ψ_E does.

We define

$$X_0 = \{v \in H^1(\Omega) \mid v(y) = 0, y \in \Gamma_O;\ \nabla v(y) \cdot \nu(y) = 0, y \in \Gamma_I\}$$
$$X_1 = X_0 \cap H^2(\Omega). \tag{31}$$

and seek solutions $(x, j_S, u_0, u_1, u_2) \in X = \mathbb{R}^{n+k_L+k_V} \times \mathbb{R} \times X_0 \times X_1^2$. The higher regularity of the Slotboom variables is needed since their gradients must have continuous generalised boundary functions [9] for the integral in (28b) to be well defined.

4. ABSTRACT DIFFERENTIAL ALGEBRAIC EQUATIONS

We formulate the PDAE (28) as an abstract DAE in the form

$$\mathcal{A}(\mathcal{D}w(t))' + bw(t) = 0, \qquad t \in \mathcal{J} = (t_0, T) \subset \mathbb{R}, \tag{32}$$

with

$$\mathcal{A} = (\ A^T\ 0\ 0\ 0\ 0\ )^T, \tag{33a}$$

$$\mathcal{D} = (\ D\ 0\ 0\ 0\ 0\ ). \tag{33b}$$

being linear and properly stated [8] and $w = (x, j_S, u_0, u_1, u_2)^T$. The operators $\mathcal{A} : Z \to Y$, $\mathcal{D} : X \to Z$ and $b : X \to Y$ are maps between the real Hilbert spaces

$$X = \mathbb{R}^{n+k_L+k_V} \times \mathbb{R} \times X_0 \times X_1 \times X_1,$$
$$Y = \mathbb{R}^{n+k_L+k_V} \times \mathbb{R} \times X_0^*(\Omega) \times L_2(\Omega) \times L_2(\Omega),$$
$$Z = \mathbb{R}^{n_C+k_L}.$$

Solutions $w : \mathcal{J} \to X$ are paths in the Hilbert space X.

The generalisation of the tractability index is a tool to analyse the time behaviour of infinite systems (abstract DAEs). Generally, the higher the index is, the more sensitive the system is with respect to perturbations of the data. We now determine the index, and investigations on the interrelationsship between the tractability index and the perturbation index for inifinte systems are upcoming.

We define $\mathcal{G}_0$ and $\mathcal{B}_0$ as

$$\mathcal{G}_0 = \mathcal{A}\mathcal{D}, \qquad \text{and} \qquad \mathcal{B}_0 = b'(w_*), \qquad t \in \mathcal{J},$$

where we understand $b'(w_*)$ as the Fréchet derivative of b at the point $w_* \in X$. We let $\mathcal{Q}$ and $\mathcal{W}$ be linear bounded operators satisfying

$$\begin{aligned} \mathcal{Q}^2 &= \mathcal{Q}, \quad \mathrm{im}\mathcal{Q} = \overline{\ker\mathcal{G}_0} \\ \mathcal{W}^2 &= \mathcal{W}, \quad \ker\mathcal{W} = \overline{\ker\mathcal{G}_0}, \end{aligned}$$

and also define

$$\mathcal{G}_1 = \mathcal{G}_0 + \mathcal{B}_0\mathcal{Q}.$$

Definition 1 ([8]) *The Abstract Differential Algebraic System (32) has ADAS index 0 when $\mathcal{G}_0$ is injective and $\overline{\mathcal{G}_0(X)} = Y$, and ADAS index 1 when*

1. dim(im$\mathcal{W}$) > 0 *and*
2. the operator $\mathcal{G}_1$ is injective and $\overline{\mathcal{G}_1(X)} = Y$.

Obviously, ker $\mathcal{G}_0 \neq \{\emptyset\}$, and we construct

$$\mathcal{B}_0 = \left(\begin{array}{cc|ccc} B & E & 0 & 0 & 0 \\ b_{21}E^T & 1 & b_{23} & b_{24} & b_{25} \\ \hline b_{31}E^T & 0 & & & \\ b_{41}E^T & 0 & & F(w_*) & \\ b_{51}E^T & 0 & & & \end{array}\right), \quad \mathcal{Q} = \begin{pmatrix} Q_0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the operators $b_{23}(\cdot) : X_0 \to \mathbb{R}$, and $b_{24}(\cdot),\ b_{25}(\cdot) : X_1 \to \mathbb{R}$, have the following actions

$$\begin{aligned} b_{23}(v) &= \int_{\Gamma_{O_1}} (\mu_n n_i \exp\big(\frac{U_{0*}}{U_T}\big)\nabla U_{1*}\cdot\nu v + \mu_p n_i \exp\big(\frac{-U_{0*}}{U_T}\big)\nabla U_{2*}\cdot\nu v)\, d\sigma, \\ b_{24}(v) &= \int_{\Gamma_{O_1}} \mu_n n_i U_T \exp\big(\frac{U_{0*}}{U_T}\big)\nu\cdot\nabla v\, d\sigma, \qquad (34) \\ b_{25}(v) &= -\int_{\Gamma_{O_1}} \mu_p n_i U_T \exp\big(\frac{-U_{0*}}{U_T}\big)\nu\cdot\nabla v\, d\sigma. \end{aligned}$$

The linearisation was carried out in the point $w_* = (x_*, j_{S*}, u_{0*}, u_{1*}, u_{2*})$ and for notational convenience we introduced

$$\psi_{E*} = hE^T x_* + e_j, \qquad U_{1*} = u_{1*} + \exp\left(\tfrac{hE^T x_* + e_j}{U_T}\right),$$
$$U_{0*} = \psi_* + \psi_{bi} + hE^T x_* + e_j, \; U_{2*} = u_{2*} + \exp\left(\tfrac{hE^T x_* + e_j}{U_T}\right).$$

We are not going to specify the functions b_{21}-b_{51} for reasons soon to be clarified. In $\mathcal{B}$, the block $F(w_*)$ produces, when evaluated, the linearised version of the semiconductor equations (28c)-(28e). We construct

$$\mathcal{G}_1 = \left(\begin{array}{cc|ccc} G_1 & E & 0 & 0 & 0 \\ b_{21}E^T Q_0 & 1 & b_{23} & b_{24} & b_{25} \\ \hline b_{31}E^T Q_0 & 0 & & & \\ b_{41}E^T Q_0 & 0 & & F(w_*) & \\ b_{51}E^T Q_0 & 0 & & & \end{array}\right) \tag{35}$$

and realise that this operator is injective and densely surjective if G_1 is non-singular and $E^T Q_0 = 0$, and if the block $F(w_*)$ is injective and densely surjective. By definition we have

$$E^T Q_0 = 0 \quad \Leftrightarrow \quad A_S^T Q_C = 0 \tag{36}$$

and after some circuit topological reasoning one sees that $A_S^T Q_C = 0$ if the two nodes of the semiconductor are either connected by a capacitive path or connected to ground by capacitive paths. We assume (36) and remind the reader that the electric network without semiconductors is described by a DAE of index 1, which is equivalent to G_1 being non-singular. In the next section, we prove that the block $F(w_*)$ is injective and densely surjective.

5. UNIQUE SOLVABILITY OF THE DRIFT DIFFUSION EQUATIONS

The introduction of the Slotboom variables allows a functional analytical approach for systems of elliptic equations to prove the injectivity and dense surjectivity of the operator $F(w_*)$. The equation

$$F(w_*)u = b \tag{37}$$

can therefore be written in divergence form

$$-\mathrm{div}(B\nabla u) + Cu = b, \tag{38}$$

where B is a positive definite matrix for all t and x. By multiplication of $v \in X_0 \times X_1^2$ and integration by parts we obtain the corresponding weak formulation

$$\int_\Omega (\nabla v)^T B \nabla u dy + \int_\Omega v^T C u dy = \int_\Omega v^T b dy. \tag{39}$$

The left-hand side is a bilinear form $a = a(u, v)$, and for each $b \in X_0^* \times [L_2(\Omega)]^2$ the right-hand side is the evaluation of a linear functional $l(\cdot) : X_0 \times X_1^2 \to \mathbb{R}$. Therefore, equation (39) is equivalent to

$$a(v, u) = l(v). \tag{40}$$

A Fredholm alternative for Gårding forms will yield the result.

Definition 2 ([9]) *A bounded, bilinear form $g : X \times X \to \mathbb{R}$ is a* Gårding form *if the embedding $X \to Y$ is continuous and the Gårding inequality*

$$g(u, u) \geq c||u||_X^2 - d||u||_Y^2 \tag{41}$$

holds for all $u \in X$ with constants $c > 0$ and d. Moreover, if the embedding $X \to Y$ is compact, then the Gårding form is regular.

Throughout this section the space X is to be understood as the solution space for the drift diffusion equations, $X = X_0 \times X_1^2$, and not as the entire solution space for the PDAE as defined in Section 3. The space X consists of functions that are zero on a subset of the boundary with surface measure greater than zero [10] and, therefore, we can use the canonical norm for $H_0^1(\Omega)^3$,

$$||u||_{1,2,0} = \left(\int_\Omega \sum_{i=0}^{2} |\nabla u_i|^2 dy \right)^{1/2}, \tag{42}$$

on it. As the space Y in Definition 2 we take $L_2(\Omega)^3$ with the norm

$$|| \ u \ ||_2 = \left(\int_\Omega \sum_{i=0}^{2} |u_i|^2 dy \right)^{1/2}. \tag{43}$$

The space X is both continuously and compactly embedded in Y[9].

We apply Young's inequality to (39) with $v = u$ and obtain

$$\begin{aligned}
\int_\Omega (\nabla u)^T B \nabla u dy + \int_\Omega u^T C u dy \geq c \int_\Omega \left(|\nabla u_0|^2 + |\nabla u_1|^2 + |\nabla u_2|^2 \right) dy \\
-d \int_\Omega \left(|u_0|^2 + |u_1|^2 + |u_2|^2 \right) dy \\
= c||u||_{1,2,0}^2 - d||u||_2^2,
\end{aligned}$$

with $c, d > 0$. Hence, a fulfils the Gårding inequality and is a regular Gårding form.

Theorem 1 ([9]) *Consider equation (40). Let $a : X \times X \to \mathbb{R}$ be a regular Gårding form on the real Hilbert space X and let $l \in X^*$. If the homogeneous equation (40) with $l = 0$ has the trivial solution $u = 0$ only, then, for each $l \in X^*$, the inhomogeneous equation (40) has a unique solution.*

We prove the uniqueness under the assumption that the point of linearisation is a solution of the original system (28). This allows us to replace the divergence terms in the first column of the matrix function C by the recombination-generation term S_*. The existence and local uniqueness of solutions of (28) for $\Omega \in \mathbb{R}$ was shown in [3]. We let $v = (\delta u_0, u_1, u_2) \in X$, with $\delta > 0$ a small parameter, and have

$$\int_\Omega \nabla \begin{pmatrix} \delta u_0 \\ u_1 \\ u_2 \end{pmatrix}^T \begin{pmatrix} \frac{\varepsilon}{q} & 0 & 0 \\ 0 & \left(n_i \mu_n U_T e^{U_{0*}/U_T}\right) & 0 \\ 0 & 0 & \left(n_i \mu_p U_T e^{-U_{0*}/U_T}\right) \end{pmatrix} \nabla \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} +$$

$$\begin{pmatrix} \delta u_0 \\ u_1 \\ u_2 \end{pmatrix}^T \begin{pmatrix} n_i e^{U_{0*}/U_T} U_{1*} + n_i e^{-U_{0*}/U_T} U_{2*} & n_i e^{-U_{0*}/U_T} & -n_i e^{U_{0*}/U_T} \\ \frac{S_*}{U_T} + \frac{\partial S_*}{\partial u_0} & \frac{\partial S_*}{\partial u_1} & \frac{\partial S_*}{\partial u_2} \\ -\frac{S_*}{U_T} - \frac{\partial S_*}{\partial u_0} & \frac{\partial S_*}{\partial u_1} & \frac{\partial S_*}{\partial u_2} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} dy$$

$$= 0.$$

We put $a = n_i \exp(U_{0*}/U_T)$ and $b = n_i \exp(-U_{0*}/U_T)$, keep the divergence terms to the left and move the rest to the right-hand side. By applying the Poincaré-Friedrich inequality on the remaining left-hand side we obtain

$$\int_\Omega [\frac{C_1 \delta \epsilon}{q} |u_0|^2 + C_1 C_2 \left(|u_1|^2 + |u_2|^2\right)] dy \leq \tag{44a}$$

$$\int_\Omega [\frac{C_1 \delta \epsilon}{q} |u_0|^2 + C_1 \mu_n a |u_1|^2 + C_1 \mu_p b |u_2|^2] dy \leq \tag{44b}$$

$$\begin{aligned} -\int_\Omega [&\delta(aU_{1*} + bU_{2*}) u_0^2 + \delta b u_0 u_1 - \delta a u_0 u_2 + \\ &(\frac{S_*}{U_T} + \frac{\partial S_*}{\partial u_0}) u_0 u_1 + \frac{\partial S_*}{\partial u_1} u_1^2 + \frac{\partial S_*}{\partial u_2} u_1 u_2 + \\ &-(\frac{S_*}{u_T} + \frac{\partial S_*}{\partial u_0}) u_0 u_2 + \frac{\partial S_*}{\partial u_1} u_1 u_2 + \frac{\partial S_*}{\partial u_2} u_2^2] \, dy \end{aligned} \tag{44c}$$

with $C_2 = \min(\inf_\Omega\{\mu_n a\}, \inf_\Omega\{\mu_p b\}) > 0$ and C_1 being the Poincaré-Friedrich constant. If $\Omega \subset (-l, l)^2$, then $C_1 = 1/2l$. We apply Young's inequality again and obtain

$$
\begin{aligned}
(44a) \leq \int_\Omega [\big(&-\delta(aU_{1*} + bU_{2*}) + \delta + |\frac{S_*}{U_T} + \frac{\partial S_*}{\partial u_0}|\big)u_0^2 \\
&+ \frac{1}{2}\big(\delta b^2 + |\frac{S_*}{U_T} + \frac{\partial S_*}{\partial u_0}| - \frac{\partial S_*}{\partial u_1} + \frac{\partial S_*}{\partial u_2}\big)u_1^2 \\
&+ \frac{1}{2}\big(\delta a^2 + |\frac{S_*}{U_T} + \frac{\partial S_*}{\partial u_0}| + \frac{\partial S_*}{\partial u_1} - \frac{\partial S_*}{\partial u_2}\big)u_2^2]dy
\end{aligned}
$$

Now, if

$$-\delta(aU_{1*} + bU_{2*}) + \delta + |\frac{S_*}{U_T} + \frac{\partial S_*}{\partial u_0}| \leq \frac{C_1 \delta \epsilon}{2q} \tag{46a}$$

$$\delta b^2 + |\frac{S_*}{U_T} + \frac{\partial S_*}{\partial u_0}| - \frac{\partial S_*}{\partial u_1} + \frac{\partial S_*}{\partial u_2} \leq C_1 C_2 \tag{46b}$$

$$\delta a^2 + |\frac{S_*}{U_T} + \frac{\partial S_*}{\partial u_0}| + \frac{\partial S_*}{\partial u_1} - \frac{\partial S_*}{\partial u_2} \leq C_1 C_2 \tag{46c}$$

for all $y \in \Omega$, we are done. Since U_{1*} and U_{2*} always remain non-negative, the inequality (46a) is fulfilled if

$$|\frac{S_*}{U_T} + \frac{\partial S_*}{\partial u_0}| \leq \frac{1}{2}(\frac{C_1 \epsilon}{q} - 2)\delta. \tag{47a}$$

The inequalities (46b) and (46c) are satisfied if

$$|\frac{\partial S_*}{\partial u_2} - \frac{\partial S_*}{\partial u_1}| \leq \frac{1}{3} C_1 C_2 \tag{47b}$$

$$\delta b^2 \leq \frac{1}{3} C_1 C_2 \tag{47c}$$

$$|\frac{S_*}{U_T} + \frac{\partial S_*}{\partial u_0}| \leq \frac{1}{3} C_1 C_2 \tag{47d}$$

$$\delta a^2 \leq \frac{1}{3} C_1 C_2 \tag{47e}$$

hold for all $y \in \Omega$. Now, we choose a δ such that

$$\delta \leq \frac{C_1}{3} \frac{\min(\inf_\Omega(\mu_n a), \inf_\Omega(\mu_p b))}{\max(\sup_\Omega a^2, \sup_\Omega b^2)} \tag{48}$$

for all $y \in \Omega$. By differentiating the recombination-generation S one sees that (47b) and (47d) can be satisfied by choosing solutions close to thermal equilibrium. Since we are primarily interested in solutions close to equilibrium due to the low voltages applied in integrated memory circuits, this restriction is not a severe one.

We have (47), fulfilled for all $y \in \Omega$, which implies

$$C \int_{\Omega} [|u_0|^2 + |u_1|^2 + |u_2|^2] dy \leq 0 \tag{49}$$

with the constant

$$C = \frac{1}{4l} \min \left\{ \frac{\delta\epsilon}{q}, \min\{\inf_{\Omega}(\mu_n a), \inf_{\Omega}(\mu_p b)\} \right\}$$

and, hence, $u = 0$. Theorem 1 now ensures the unique solvability of (37), and thereby $\mathcal{G}_1$ is injective and densely solvable, thus, the PDAE is of ADAS index one.

In the next section we summarise our result and the assumptions under which it is valid.

6. RESULTS

By coupling an electric network DAE of index 1 without semiconductors with the drift diffusion equations modelling the semiconductors, we obtain a PDAE or abstract DAE of the form (28) having ADAS index 1 if the applied voltages are low. This result is valid under the assumptions that the nodes of the semiconductor are either connected by a capacitive path or connected to ground by such paths.

In [11] it is shown that also the instationary version of the drift diffusion equations yields an index 1 PDAE.

We conclude our discussion by giving an example: the frequency multiplier.

7. THE FREQUENCY MULTIPLIER

A simple example of a circuit that fulfils the index 1 conditions and the condition (36) on the position of the semiconductor is the frequency multiplier (Figure 1).

The controlled voltage source produces an alternating current in the left part of the circuit with frequency f_1, which is the eigenfrequency of the left oscillator, $f_1 = (L_1 C_1)^{-1}$. The non-linear behaviour of the diode generates a mixture of frequencies in the right part of the circuit. The resistances satisfiy $R_S = R_O$ and the right oscillator resonates on the double frequency $(f_2 = 2f_1)$, and all the other frequencies are damped.

The network without the semiconductor branch is modelled as a DAE with the

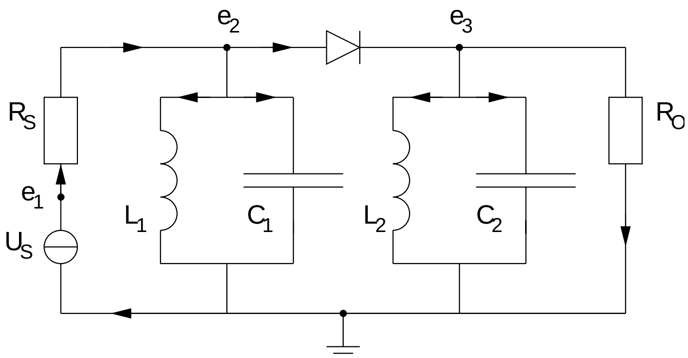

Fig. 1. A frequency multiplier

incidence matrices

$$A_C = A_L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_R = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_V = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \tag{50}$$

the projectors

$$Q_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_C = I - Q_C, \tag{51}$$

and the constant component matrices

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}, \quad L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \quad G = \begin{pmatrix} R_S & 0 \\ 0 & R_O \end{pmatrix}. \tag{52}$$

According to the topological criteria mentioned in Section 2 the circuit is index 1 and we see that the matrix

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & C_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{53}$$

is non-singular.

The nodes of the semiconductor are connected by a capacitive path and we have

$$A_S^T Q_C = 0 \quad \text{with} \quad A_S = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{54}$$

If we model the behaviour of the semiconductor with the drift diffusion equations we end up with a system of the form (28) with ADAS index 1.

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